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Nonlinear vibration analysis of initially stressed thin laminated rectangular plates on elastic foundations

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Abstract

Studies are made on the elastic behaviour of laminated rectangular thin plates on elastic foundations with combined lateral and compressive in-plane forces. The von Kármán's large deflection equations for generally laminated elastic plates are derived in terms of stress function and deflection function. A deflection function satisfying the geometric boundary conditions is assumed and a stress function is then obtained after solving the compatibility equation. The modified Galerkin's method is applied to the governing nonlinear partial differential equation to obtain the nonlinear ordinary differential equation of motion (modal equation). Procedure for exact integration of the modal equation is described. Numerical results of simply supported as well as clamped square plates are presented. It is found that the nonlinear frequency increases with the amplitude for the applied lateral and compressive inplane forces. Analytical expressions for the constants in the modal equation are provided to use for any lay-up sequence.

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1. Introduction

Laminated composite plates and shells are being used in aerospace and other engineering applications as lightweight high-strength structural components. Two types of nonlinearities are most commonly encountered in plate problems. The geometric nonlinearity arises due to large deformation and material nonlinearity is used to deal with nonlinear material having stress–strain behavior that is not linear. For nonlinear analysis of thin laminated anisotropic plates, the governing equations consist of a system of nonlinear partial differential equations of eighth order in terms of three displacement components (u, v, and w) or in terms of the transverse deflection (w) and force function (ϕ). The nonlinear partial equations governing composite laminates of arbitrary geometries and boundary conditions cannot be solved exactly. Approximate analytical solutions to the large–deflection theory (in von Kármán sense) of laminated composite plates were obtained by utilizing Rayleigh–Ritz method, Galerkin method, perturbation method, and the double-series method [1–15]. The application of orthotropic plate structures as basis for nuclear reactors, aircraft runways, building foundation slabs, indoor sports, floors etc. is becoming wide spread. Therefore, the study of the nonlinear analysis of thin laminated rectangular plates resting on elastic foundations is of importance in the optimum design of these structures. The use of large finite element programs, which are capable of handling virtually any degree of complexity, is cumbersome, costly, and time consuming. It is preferable to use continuum methods, if closed-form solution methods for such a system are not possible. Simple continuum solution methods can provide not only a check against the computer finite element model, but also a means by which the effect of a parameter change on a system can be readily gauged, which is useful in the design process.

This paper examines the large-amplitude vibration of clamped as well as simply supported laminated thin rectangular plates on elastic foundations with combined lateral and compressive in-plane forces. The von Kármán's large deflection equations for generally laminated elastic plates are derived in terms of stress function and deflection function. A deflection function satisfying the geometric boundary conditions is assumed and a stress function is then obtained after solving the compatibility equation. The modified Galerkin's method is applied to the governing nonlinear partial differential equation to obtain the nonlinear ordinary differential equation of motion (modal equation). Procedure for exact integration of the equation of motion is described. Analytical expressions for the constants in the equation of motion are provided to use for any lay-up sequence. It is found that the nonlinear frequency increases with the amplitude for the applied lateral and compressive in-plane forces.

2. Formulation

The governing equations are based on von Kármán's elastic thin plate theory assumptions. Three additional assumptions are added. First, there is no slip between the adjacent layers of the laminated plate. Second, rotatory inertia and transverse shear deformation effects are neglected. Third, kinematic relations, $(\partial u/\partial x)^2$ and $(\partial v/\partial y)^2$ are neglected as compared with $(\partial w/\partial x)^2$ and $(\partial w/\partial y)^2$ terms.

A thin rectangular plate of length a in the x direction, width b in the y direction and thickness h, in the z direction is considered. The mid surface of the un-deformed plate, which contains the x- and y-axis, is in the reference plane ($z=0$). The plate is thin (i.e., $h \ll a,\ h \ll b$) and it is constructed of an arbitrary number of anisotropic layers of arbitrary arrangement and thickness. Stress resultants and moments provide a simple means of dealing with laminated behavior.

Nonlinear equations of motion of generally laminated plates are

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0, \tag{1}$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0, \tag{2}$$

$$\begin{aligned} &\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x\,\partial y} + \frac{\partial^2 M_y}{\partial y^2} + N_x \frac{\partial^2 w}{\partial x^2} \\ &\quad + 2N_{xy}\frac{\partial^2 w}{\partial x\,\partial y} + N_y \frac{\partial^2 w}{\partial y^2} + q + q_{ef} = \sum \rho_i h_i \frac{\partial^2 w}{\partial t^2}, \end{aligned} \tag{3}$$

where ρ_i and h_i are the density and thickness of the ith layer,

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \varepsilon \\ \chi \end{Bmatrix}. \tag{4}$$

The components in the matrices of the stress and moment resultants, viz., $N = \{N_x, N_y, N_{xy}\}^{\mathrm{T}}$ and $M = \{M_x, M_y, M_{xy}\}^{\mathrm{T}}$ are defined as

$$(N_k, M_k) = \int_{-h/2}^{h/2} (1, z)\sigma_k \,\mathrm{d}z \quad (k = x, y, xy).$$

N_x, N_y, N_{xy} are membrane forces per unit length, M_x, M_y, M_{xy} are the bending and twisting moments per unit length. The elements A_{ij}, B_{ij} and D_{ij} $(i,j = 1, 2, 6)$ in the 3×3 symmetric matrices of A, B, and D in Eq. (4) are defined as

$$(A_{ij}, B_{ij}, D_{ij}) = \int_{-h/2}^{h/2} (1, z, z^2) Q_{ij} \,\mathrm{d}z \quad (i,j = 1, 2, 6).$$

The elements A_{ij}, B_{ij}, and D_{ij}, are, respectively, the membrane rigidities, coupling rigidities, and flexural rigidities of the plate. Q_{ij} are the reduced stiffness coefficients, which can be related to the more familiar engineering moduli by $\sigma = \{Q_{ij}\}\varepsilon$. The components in the matrices of in-plane stress and strains are $\sigma = \{\sigma_x, \sigma_y, \sigma_{xy}\}^{\mathrm{T}}$ and $\varepsilon = \{\varepsilon_x, \varepsilon_y, \varepsilon_{xy}\}^{\mathrm{T}}$. The matrices of strains (ε) and curvature changes (χ) are written by considering von Kármán type of geometric nonlinearity, as

$$\varepsilon = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} + \dfrac{1}{2}\left(\dfrac{\partial w}{\partial x}\right)^2 \\ \dfrac{\partial v}{\partial y} + \dfrac{1}{2}\left(\dfrac{\partial w}{\partial y}\right)^2 \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} + \dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} \end{Bmatrix}, \tag{5}$$

$$\kappa = \begin{Bmatrix} \chi_x \\ \chi_y \\ \chi_{xy} \end{Bmatrix} = -\begin{Bmatrix} \dfrac{\partial^2 w}{\partial x^2} \\ \dfrac{\partial^2 w}{\partial y^2} \\ 2\dfrac{\partial^2 w}{\partial x\,\partial y} \end{Bmatrix}, \tag{6}$$

u, v, w are the displacements at the reference plane ($z = 0$). ε_x, ε_y, ε_{xy} are the reference surface strains. $\chi_x, \chi_y, \chi_{xy}$ are the curvature changes.

Eq. (4) is rewritten as

$$\begin{Bmatrix} \varepsilon \\ M \end{Bmatrix} = \begin{bmatrix} A^* & B^* \\ -(B^*)^{\mathrm{T}} & D^* \end{bmatrix} \begin{Bmatrix} N \\ \kappa \end{Bmatrix}, \tag{7}$$

where $[A^*] = [A]^{-1}$, $[B^*] = -[A]^{-1}[B]$, and $[D^*] = [D] + [B][B^*]$.

The Airy stress function φ, which satisfies Eqs. (1) and (2) is defined by

$$N_x = \frac{\partial^2 \varphi}{\partial y^2}, \tag{8a}$$

$$N_y = \frac{\partial^2 \varphi}{\partial x^2}, \tag{8b}$$

$$N_{xy} = -\frac{\partial^2 \varphi}{\partial x\,\partial y}. \tag{8c}$$

In general, there are two types of foundations. The "attached foundation", in which the plate cannot separate from the elastic medium and the intensity of reaction from the medium is proportional to the deflection of the plate whether it buckles into or away from the foundation. The reaction, when expressed as force per unit area per unit deflection, is the "modulus of foundation". The second type of foundation is the "detached foundation". When the plate buckles into waves, the deflection in the opposite direction causes the plate to pull away from the foundation without any concomitant reaction. In both these conditions of support, the assumption that there is a direct linear relationship between the deflection at any point and the reaction of the medium at that point is of course the simplest that can be made. It is known as Winkler-type foundation. The elastic foundation introduces a transverse distributed force on the plate given by

$$q_{ef} = -\left(kw + k_1 w^3 - g\frac{\partial^2 w}{\partial x^2} - g\frac{\partial^2 w}{\partial y^2}\right), \tag{9a}$$

where k is the Winkler foundation parameter, k_1 is the nonlinear Winkler foundation parameter and g is the shear parameter of Pasternak model foundation.

The transverse load $q(x, y)$ is defined as

$$q(x, y) = q_{\max} f(x, y), \tag{9b}$$

where $q_{\max}$ is the maximum load and $|f(x, y)| \leqslant 1$, $\forall(x, y)$ inside the boundary of the plate. In the present analysis, the problems are solved for the case of uniformly distributed transverse load (i.e., $f(x, y) = 1$).

The compatibility equation is derived from relation (5) as

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} - \frac{\partial^2 \varepsilon_{xy}}{\partial x\, \partial y} = \left(\frac{\partial^2 w}{\partial x\, \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}. \tag{10}$$

Using Eqs. (7)–(9) in Eqs. (3) and (10), one obtains

$$\sum \rho_i h_i \frac{\partial^2 w}{\partial t^2} + L_1(w) + L_3(\varphi) - L(\varphi, w) - q_{\max} + \left(kw + k_1 w^3 - g\frac{\partial^2 w}{\partial x^2} - g\frac{\partial^2 w}{\partial y^2}\right) = 0, \tag{11}$$

$$L_2(\varphi) - L_3(w) - \left(\frac{\partial^2 w}{\partial x\, \partial y}\right)^2 + \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial^2 y} = 0, \tag{12}$$

where the differential operators are

$$L_1 = D_{11}^* \frac{\partial^4}{\partial x^4} + 4D_{16}^* \frac{\partial^4}{\partial x^3 \partial y} + 2(D_{12}^* + 2D_{66}^*)\frac{\partial^4}{\partial x^2 \partial y^2} + 4D_{26}^* \frac{\partial^4}{\partial x \partial y^3} + D_{22}^* \frac{\partial^4}{\partial y^4},$$

$$L_2 = A_{22}^* \frac{\partial^4}{\partial x^4} - 2A_{26}^* \frac{\partial^4}{\partial x^3 \partial y} + (2A_{12}^* + A_{66}^*)\frac{\partial^4}{\partial x^2 \partial y^2} - 2A_{16}^* \frac{\partial^4}{\partial x\, \partial y^3} + A_{11}^* \frac{\partial^4}{\partial y^4},$$

$$\begin{aligned} L_3 = {} & B_{21}^* \frac{\partial^4}{\partial x^4} + (2B_{26}^* - B_{61}^*)\frac{\partial^4}{\partial x^3 \partial y} + (B_{11}^* + 2B_{22}^* - 2B_{66}^*)\frac{\partial^4}{\partial x^2 \partial y^2} \\ & + (2B_{16}^* - B_{62}^*)\frac{\partial^4}{\partial x\, \partial y^3} + B_{12}^* \frac{\partial^4}{\partial y^4}, \end{aligned}$$

$$L(\phi, w) = \frac{\partial^2 \phi}{\partial y^2}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 \phi}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - 2\frac{\partial^2 \phi}{\partial x\, \partial y}\frac{\partial^2 w}{\partial x\, \partial y}.$$

Therefore, Eqs. (11) and (12) are two coupled governing equations of arbitrarily laminated thin plates on elastic foundations.

3. Analysis

The large-amplitude vibrations of simply supported and clamped anisotropic rectangular plates are examined here by applying the Galerkin's method.

Boundary conditions for simply supported rectangular plates are

$$w = 0,\ M_x = 0 \quad \text{at } x = 0, a, \tag{13}$$

$$w = 0,\ M_y = 0 \quad \text{at } y = 0, b. \tag{14}$$

For clamped plates the boundary conditions are

$$w = 0,\ \frac{\partial w}{\partial x} = 0 \quad \text{at } x = 0, a, \tag{15}$$

$$w = 0,\ \frac{\partial w}{\partial y} = 0 \quad \text{at } y = 0, b. \tag{16}$$

The in-plane boundary conditions are

$$\int_0^a (N_y)_{y=0,b}\, \mathrm{d}x = -P_y, \tag{17a}$$

$$\int_0^a (N_{xy})_{y=0,b}\, \mathrm{d}x = 0, \tag{17b}$$

$$\int_0^b (N_x)_{x=0,a}\, \mathrm{d}y = -P_x, \tag{17c}$$

$$\int_0^b (N_{xy})_{x=0,a}\, \mathrm{d}y = 0. \tag{17d}$$

Here P_x and P_y are compressive loads applied along x and y directions, respectively.

The system of equations (11) and (12) in terms of the transverse deflection (w) and force function (φ), are to be solved in conjunction with the boundary conditions (13)–(17). A deflection function (for w) satisfying the geometric boundary conditions of the plate is assumed. A stress function (φ) is then obtained from the compatibility equation (12). Galerkin's method is applied to the governing nonlinear partial differential equation to yield a second-order nonlinear differential equation of motion in time variable. The details of the solution of the problem are briefly described below.

The transverse supporting conditions given in (13)–(16) are satisfied by assuming the deflection functions of the laminate corresponding to the (m, n) in the separable form as

$$w = W_{mn}(t) \sin(\alpha_m x) \sin(\beta_n y) \tag{18}$$

for simply supported boundary conditions, and

$$w = W_{mn}(t) \sin^2(\alpha_m x) \sin^2(\beta_n y), \tag{19}$$

for clamped boundary conditions. Here $\alpha_m = m\pi/a$, $\beta_n = n\pi/b$, m and n are the axial and transverse wave numbers, respectively.

Substituting the transverse deflection, w into Eq. (12), and solving utilizing the in-plane boundary conditions, the stress function (φ) is obtained as

$$\varphi = W_{mn}\varphi_1(x,y) + W_{mn}^2\varphi_2(x,y) - \frac{1}{2}\left[\frac{P_x y^2}{b} + \frac{P_y x^2}{a}\right]. \tag{20}$$

The expressions for the functions φ_1 and φ_2 are given in Appendix A.

Upon substitution of w and φ into Eqs. (13) and (14), the force boundary conditions can not be satisfied when B^* does not equal zero. As in Ref. [7], the modified Galerkin's method is adopted here wherein the residues on boundaries are minimized. In the case of simply supported rectangular plates assuming $\bar{w} = \sin\alpha_m x \sin\beta_n y$, the residual force and moment have the following relation:

$$\int_0^a\int_0^b L^R(\varphi,w)\bar{w}\mathrm{d}x\,\mathrm{d}y + \int_0^b \left(M_x\frac{\partial\bar{w}}{\partial x}\right)_{x=0}\mathrm{d}y + \int_0^b \left(M_x\frac{\partial\bar{w}}{\partial x}\right)_{x=a}\mathrm{d}y$$
$$+ \int_0^a M_y\left(\frac{\partial\bar{w}}{\partial y}\right)_{y=0}\mathrm{d}x + \int_0^a M_y\left(\frac{\partial\bar{w}}{\partial y}\right)_{y=b}\mathrm{d}x = 0, \tag{21}$$

where $L^R(\varphi,w)$ is residual force.

Letting $L^N(\varphi,w) = L^R(\varphi,w) + L(\varphi,w)$, Eq. (11) then becomes

$$\sum \rho_i h_i \frac{\partial^2 w}{\partial t^2} + L_1(w) + L_3(\varphi) - L^N(\varphi,w) - q_{\max} + (kw + k_1 w^3 - g\frac{\partial^2 w}{\partial x^2} - g\frac{\partial^2 w}{\partial y^2}) = 0. \tag{22}$$

Applying Galerkin's method to Eq. (22), the modal equation is then obtained as

$$\sum \rho_i h_i \frac{\mathrm{d}^2 W_{mn}}{\mathrm{d}t^2} + (\alpha - \alpha_p)W_{mn} + \beta W_{mn}^2 + \gamma W_{mn}^3 - \delta q_{\max} = 0. \tag{23}$$

Similar equation is obtained for the case of clamped rectangular plates by substituting w and φ into Eq. (11) and applying the Galerkin's method. The constants α, α_p, β, γ, and δ in the equation of motion (23) for both simply supported rectangular plates and clamped rectangular plates are defined in Appendix B.

In the present dynamic formulation, the load versus frequency curve (namely the eigencurve) of the plate is essential for studying the stability of the equilibrium position of the plate as well as for the large deflection (post-buckling) analysis of the plate. Stability loads are those loads at which the eigencurve meets the load axis (the frequency, $\omega = 0$).

Defining $\zeta = W_{mn}/h$ and $\tau = \omega t$, the modal equation (23) is written in the form

$$\sum \rho_i h_i \omega^2 \ddot{\zeta} + (\alpha - \alpha_p)\zeta + \beta h\zeta^2 + \gamma h^2\zeta^3 = \frac{\delta q_{\max}}{h}, \tag{24}$$

where ω is the nonlinear frequency and over-dot denotes differentiation with respect to τ.

3.1. Post-buckling load–deflection relation

Post-buckling load–deflection relation for the problem in the absence of transverse uniformly distributed load (i.e., $q_{\max} = 0$) can be obtained from Eq. (24) by substituting $\omega = 0$ as

$$\frac{P_x \alpha_m^2}{b} + \frac{P_y \beta_n^2}{a} = C\alpha[1 + \delta_1 \zeta + \delta_2 \zeta^2]. \tag{25}$$

For the case of simply supported plates the constant, $C = 1$, whereas in the case of clamped plates $C = 4/3$. By neglecting the contribution of ζ and ζ^2 terms in Eq. (25), the buckling load can be determined using

$$\frac{P_x \alpha_m^2}{b} + \frac{P_y \beta_n^2}{a} = C\alpha. \tag{26}$$

Eq. (26) may be used to determine the onset of buckling under a number of different types of loading. For example, when $P_y = 0$, critical axial buckling load corresponds to

$$P_{xc} = \frac{C\alpha b}{\alpha_m^2}. \tag{27}$$

The corresponding expression may be derived when $P_x = 0$, for the transverse buckling load,

$$P_{yc} = \frac{C\alpha a}{\beta_n^2}. \tag{28}$$

If the load P_y, for example, is regarded as a constant preload, with only P_x allowed to vary, the corresponding form of Eq. (26) is

$$P_{xc} = \left(C\alpha - \frac{P_y \beta_n^2}{a} \right) \frac{b}{\alpha_m^2}. \tag{29}$$

As would be expected, Eq. (29) shows tensile ($-P_y$) to have a stabilizing effect, while compressive P_y is destabilizing. Alternatively, the load P_y may be assumed to vary proportionately with P_x until buckling takes place, with

$$P_x = P, \quad P_y = K_b P \tag{30}$$

in Eq. (26) gives, at the buckling

$$P_c = C\alpha \left(\frac{\alpha_m^2}{b} + \frac{K_b \beta_n^2}{a} \right)^{-1}. \tag{31}$$

Using Eqs. (25),(30) and (31), the load–deflection relation can be written in nondimensional form as

$$\frac{P}{P_c} = 1 + \delta_1 \zeta + \delta_2 \zeta^2. \tag{32}$$

Knowing the applied load (P) in the post-buckling range, Eq. (32) can be solved for ζ. Thus, the unknown maximum deflection-to-thickness ratio corresponding to the post-buckling load can be determined. For the specified maximum deflection the post-buckling load can be determined directly from Eq. (32).

3.2. Maximum transverse load–deflection relation

The maximum transverse load–deflection relation of the problem in the absence of compressive loads (i.e., $P_x = P_y = 0$) can be obtained from Eq. (24) by substituting $\omega = 0$ as

$$\frac{Q_{NL}}{Q_{LF}} = \zeta + \delta_1\zeta^2 + \delta_2\zeta^3, \tag{33}$$

where the nonlinear load parameter, $Q_{NL} = q_{\max}b^4/E_{22}h^4$; and the linear load factor, $Q_{LF} = \alpha b^4/\delta E_{22}h^3$. The load parameter Q_{NL} can be evaluated directly from Eq. (33) by specifying the values of ζ. For the specified value of Q_{NL}, ζ can be obtained by solving the cubic equation (33).

3.3. Frequency–amplitude relation

The frequency–amplitude relation for the unloaded laminated thin rectangular plates on elastic foundation is obtained here from the modal equation (24)

$$\omega^2\ddot{\zeta} + \omega_L^2 f(\zeta) = 0, \tag{34}$$

where the restoring force function, $f(\zeta) = \zeta + \delta_1\zeta^2 + \delta_2\zeta^3 = 0$; $\omega_L = \sqrt{\alpha/\sum \rho_i h_i}$, is the linear frequency of the unloaded rectangular plate.

The initial conditions for Eq. (34) are

$$\zeta = \zeta_s, \quad \dot{\zeta} = 0 \quad \text{at } t = 0. \tag{35}$$

Here ζ_s is the ratio of the maximum amplitude of the transverse deflection and the thickness of the plate.

The restoring force function $f(\zeta)$ in the equation of motion (34), is a cubic polynomial which is of Duffing type or a combination of quadratic and cubic terms. If $\delta_1 = 0$, then $f(\zeta)$ becomes an odd function, and the magnitudes of maximum positive and negative amplitudes in the periodic motion will be equal. In the case of mixed-parity (i.e., $\delta_1 \neq 0$), $f(\zeta)$ is a nonodd function and the magnitudes of maximum positive and negative amplitudes in the periodic motion will be different. Hence, for nonodd function $f(\zeta)$ the behavior of oscillations is different for the positive and negative amplitudes. That means the frequency values for the specified maximum positive and negative amplitudes having the same magnitude, will be different.

The relationship between the maximum positive and negative amplitudes, viz. ζ_+ and ζ_-, can be found by equating the potential energies in either position, i.e. from

$$I(\zeta_+) = I(\zeta_-), \tag{36}$$

where

$$I(\xi) = \int_0^{\xi} f(\eta)d\eta = \frac{\xi^2}{2}\left(1 + \tfrac{2}{3}\delta_1\xi + \tfrac{1}{2}\delta_2\xi^2\right). \tag{37}$$

Multiplying Eq. (34) by $\mathrm{d}\zeta/\mathrm{d}\tau$ and integrating,

$$\tfrac{1}{2}\omega^2\left(\frac{\mathrm{d}\zeta}{\mathrm{d}\tau}\right)^2 + \omega_L^2\{I(\zeta) - I(\zeta_-)\} = 0. \tag{38}$$

The initial conditions used while integrating Eq. (34) to obtain (38) are

$$\zeta = \zeta_-, \quad \dot{\varsigma} = 0 \quad \text{at } t = 0. \tag{39}$$

The solution curve on the $\zeta - \dot{\zeta}$ plane is referred to as the integral curve or the phase trajectory. In the periodic motion of the system, the solution curve on the $\zeta - \dot{\zeta}$ plane is a closed trajectory [16]. If $\delta_1 = 0, f(\zeta)$ is an odd function and $I(\xi)$ is an even function. The solution curve of Eq. (38) will be symmetric on ζ and $\dot{\zeta}$ axis. If $\delta_1 \neq 0$, then $f(\zeta)$ becomes an odd function. The solution curve of Eq. (37) will be symmetric only on the ζ axis.

Integrating Eq. (38) from $\tau = 0$ to π, one gets

$$\frac{\omega_L}{\omega} = \frac{1}{\sqrt{2}\pi}\int_{\zeta_-}^{\zeta_+} \frac{\mathrm{d}\zeta}{\sqrt{I(\zeta_-) - I(\zeta)}}. \tag{40}$$

For a specified maximum positive amplitude-to-thickness ratio ζ_+ the corresponding maximum negative amplitude-to-thickness ratio ζ_- is obtained from Eq. (36) and vice versa. By substituting ζ_- and ζ_+ in Eq. (40) the nonlinear frequency ω is obtained. The integrand in Eq. (40) has poles at the end of integration (i.e., at $\zeta = \zeta_-$ and ζ_+), which may adversely affect the accuracy of an integration rule. Hence, the integrand in Eq. (40), is modified by using

$$\zeta = \zeta_1 + \zeta_2 \cos\left\{\tfrac{1}{2}\pi(1 + \xi)\right\}. \tag{41}$$

That eliminates the singularities and yields a form

$$\frac{\omega}{\omega_L} = \left(\frac{1}{2}\int_{-1}^{1} \frac{\mathrm{d}\zeta}{\sqrt{c_0 + c_1\zeta + c_2\zeta^2}}\right)^{-1}, \tag{42}$$

where

$$c_0 = 1 + \tfrac{4}{3}\delta_1\zeta_1 + \tfrac{1}{2}\delta_2(3\zeta_1^2 + \zeta_2^2), \quad c_1 = \tfrac{2}{3}\delta_1 + \delta_2\zeta_1, \quad c_2 = \tfrac{1}{2}\delta_2,$$

$$\zeta_1 = \tfrac{1}{2}(\zeta_+ + \zeta_-),\ \zeta_2 = \tfrac{1}{2}(\zeta_+ - \zeta_-).$$

The negative amplitude-to-thickness ratio ζ_- corresponding to the positive amplitude-to-thickness ratio ζ_+ obtained from Eq. (36) is

$$\zeta_- = \frac{(s_1 + s_2 - a_2)}{a_3}, \tag{43}$$

where

$$s_{1,2} = \sqrt{r \pm \sqrt{q^3 + r^3}}, \quad q = a_3a_1 - a_2^2,\ r = \tfrac{1}{2}(3a_3a_2a_1 - a_3^2a_0) - a_2^2,$$

$$a_3 = c_2, \quad a_2 = \tfrac{1}{3}\left(\tfrac{2}{3}\delta_1 + a_3\zeta_+\right), \quad a_1 = \tfrac{1}{3} + a_2\zeta_+ \quad \text{and } a_0 = 3a_1\zeta_+.$$

A 10-point Gaussian rule was adopted while evaluating the integral in Eq. (42).

3.4. Frequency–static deflection relation

From the point of view of practical application, approximate methods of determination of the fundamental mode of elastic plates could be helpful. Jones [17] examined the applicability of a frequency-static deflection relation,

$$\rho h \omega_L^2 \left[\frac{W_{\max}}{q_{\max}} \right] = C_1 (\cong 1.630). \tag{44}$$

For plates of various geometry and boundary conditions, this relation is obtained from the expression for the fundamental frequency (ω_L) of a clamped elliptical plate [18], and the maximum deflection ($W_{\max}$) of the same plate under a uniformly distributed load ($q_{\max}$). Here, ρ is the mass per unit area and h is the plate thickness. Jones cautioned that this relation may be inappropriate if any portion of the plate boundary is freely supported. Maurizi et al. [19] have made a comparative study of various existing expressions [17,19,20] to obtain the fundamental frequency for the case of a clamped elliptical plate. Sundararajan [21] also presented a similar type of relation. This relation is based on Rayleigh's method. The fundamental mode of a rectangular plate is approximated by the deflection functions of beams subjected to uniformly distributed loads. He analyzed the plates considering all edges simply supported, two opposite edges simply supported and the others clamped, and, two opposite edges simply supported with third edge clamped and the fourth edge free. The values of the constant (C_1) in Eq. (44) for these plate configurations found by him are 1.723, 1.613, 1.667 and 1.978, respectively. The variation in the constant values is mainly due to geometry of the plate and boundary conditions. Although the simple approximate expression as suggested by Jones [17] and Sundararajn [21], are good estimates for various plate configurations, there is no formal derivation of the frequency static deflection relation for a plate of arbitrary shape and complex boundary conditions. Radhakrishnan et al. [22] have examined the possibility of such frequency–static displacement relations and proposed a methodology for estimating the fundamental frequency of plate through its static deflections under a uniformly distributed load without the associated eigenvalue problem being solved. They suggested to use the mode shape of the plate proportional to its static deflection under uniformly distributed load, for evaluating the constant C_1 and determining the fundamental frequency (ω_L) from Eq. (44). Relation (44) is verified for isotropic plates. The solutions for simply supported and clamped thin laminated rectangular plates presented in this paper are verified with relation (44) by replacing ρh as $\sum \rho_I h_i$, $W_{\max}$ as W_{11} and C_1 as δ. For simply supported rectangular plates the constant, $C_1 = 16/\pi^2$ ($\sim$1.6207), and for clamped case $C_1 = 16/9$. The values of the constant are found to be in good agreement with those of Sundararajan [21].

3.5. Nonlinear vibrations of initially stressed plates

The equation of motion for nonlinear vibrations of initially stressed plates from Eq. (24) is written in the form

$$\left(\frac{\omega}{\omega_L} \right)^2 \ddot{\zeta} + \left(1 - \frac{P}{P_c} \right) \zeta + \delta_1 \zeta^2 + \delta_2 \zeta^3 = \frac{Q_{NL}}{Q_{LF}}, \tag{45}$$

where $P/P_c = \alpha_P/\alpha$ and $Q_{NL}/Q_{LF} = \delta q_{\max}/(\alpha h)$. The initial conditions for Eq. (45) are the same as those given in Eq. (35). In the present analysis, the applied compressive load (P) is less than the buckling load (P_c) and the transverse uniformly distributed load ($q_{\max}$) is for the amplitude ratio (ζ_0) less than the specified amplitude ratio (ζ_s). The equation of motion (45) with the initial condition (35), is solved numerically as in Section 3.3.

4. Results and discussion

This paper examines the elastic behavior of laminated rectangular thin plates with moderately large deflection, post-buckling and nonlinear vibration. Numerical results of simply supported as well as clamped rectangular plates are presented for the dimensionless constants of elastic foundation parameters:

$$K = \frac{ka^4}{D_{11}}, \quad G = \frac{ga^2}{D_{11}} \quad \text{and} \quad K_1 = \frac{k_1 a^4 h^2}{D_{11}}.$$

Table 1 gives the comparison of buckling loads of clamped square plates made of different materials without elastic foundations. The analytical results are found to be reasonably in good agreement with test results [23]. For isotropic simply supported square plates without-elastic foundations, Little [9] represented the transverse displacement (w) by nine double Fourier series (which satisfies all the out-of-plane boundary conditions) and substituted into the von Kármán compatibility equation for obtaining the force function (ϕ). For the specified value of $\zeta_s = 1$, the nonlinear load parameter, Q_{NL} reported by Little [9] was 29.4, whereas the present analysis based upon a one-mode Galerkin approximation gives the result from Eq. (33) as 29.52. For the specified value of $\zeta_s = 2$, Little reported the value of Q_{NL} as 99.4, whereas the present analysis gives the result as 104.1. The mechanical properties in the unidirectional laminate assumed for glass-epoxy material are $E_{11}/E_{22} = 3$, $G_{12}/E_{22} = 0.5$ and $\nu_{12} = 0.25$. For the simply supported unidirectional glass-epoxy square plate, the nonlinear load parameter Q_{NL} is 50 for $\zeta_s = 1.1315$, which is reported in Table 5.3 of the textbook/monograph of Chia [10] by considering the first four terms in each of the truncated generalized Fourier series for ϕ and w. Based upon a one-mode Galerkin approximation, the present analysis yields the result of Q_{NL} for $\zeta_s = 1.1315$ as 54.27. These two approximate solutions are reasonably in good agreement with each other. Table 2 gives the fundamental linear frequency (ω_L) of two layered cross-ply square plates.

Table 1
Comparison of buckling loads (P_{xc}) for clamped square plates of different materials ($a = b = 254$ mm)

Material	Moduli (GPa)			Poission's ratio, ν_{12}	Thickness (mm)	Buckling load, P_{xc} (KN)	
	E_{11}	E_{12}	G_{12}			Test [23]	Eq. (27)
Aluminum	72.4	72.4	27.2	0.33	3.23	95.6	94.2
Steel	200	200	76.9	0.30	3.20	222.4	248.8
Boron/epoxy	213.7	18.62	0.52	0.28	2.92	68.9	78.9
$[0/90]_{5s}$					2.59	61.8	55.0
					2.31	45.8	39.1

Table 2
The fundamental linear frequency, $\Omega_L \left(\equiv \omega_L b^2 \sqrt{\sum \rho_i h_i/(E_{22}h^3)}\right)$ of two layered cross-ply square plates ($K = K_1 = G = 0$)

Material	$\frac{E_{11}}{E_{22}}$	$\frac{G_{12}}{E_{22}}$	v_{12}	Linear frequency, Ω_L			
				Simply supported		Clamped	
				Chia and Prabhakara [24]	Present study	Chia [10]	Present study
Glass-epoxy	3	0.50	0.25	6.8143	6.8143	12.9760	13.3546
Boron-epoxy	10	0.33	0.30	7.5638	7.6153	15.7775	16.2055
Graphite-epoxy	40	0.50	0.25	11.1641	11.1640	24.0332	24.5454

Table 3
The constants in the equation of motion for a two-layered glass-epoxy cross-ply square plate on elastic foundations ($\delta_1 = 0$)

Case	Foundation parameters			δ_2	$\frac{bP_c}{E_{22}h^3}$	Q_{LF}	$\Omega_L = \omega_L b\sqrt{\sum \rho_i h_i/(E_{ZZ}h^3)}$
	K	K_1	G				
Simply supported square plate							
1	0	0	0	0.5272	4.7048	28.64	6.8143
2	50	0	0	0.4456	5.5671	33.89	7.4125
3	50	0	25	0.1762	14.0778	85.71	11.7873
4	50	25	0	0.4891	5.5671	33.89	7.4125
5	50	25	50	0.1206	22.5884	137.52	14.9311
Clamped square plate							
1	0	0	0	0.3054	24.0938	100.32	13.3546
2	50	0	0	0.2915	25.2436	105.11	13.6696
3	50	0	25	0.1822	40.3736	168.11	17.2873
4	50	25	0	0.3036	25.2436	105.11	13.6696
5	50	25	50	0.1381	55.5036	231.10	20.2694

For a simply supported isotropic square plate, the constants in the equation of motion (34) are $\delta_1 = 0$ and $\delta_2 = 0.34125$, which are found to be in good agreement with those from the expressions of Yamaki [25]. For the maximum amplitude ratio, $\zeta_s = \zeta_+ = 1$ the frequency ratio (ω/ω_L) from the graphical result of Yamaki [25] gives 1.12, while the present analysis from Eq. (42) gives 1.1197. For a clamped isotropic square plate, the constants in the equation of motion (34) are $\delta_1 = 0$ and $\delta_2 = 0.22736$. The frequency ratio (ω/ω_L) for the specified amplitude ratio $\zeta_s = \zeta_+ = 2$, is found from Eq. (42) is 1.2925, whereas it is 1.2987 from the graphical results of Yamaki [25]. Table 3 gives the constants in the equation of motion (34) for a two layered glass-epoxy cross-ply plate on elastic foundation. It is noted from the results in Table 4 that the nonlinear frequency increases with the amplitude for the specified elastic foundation parameter. Regarding the analysis for moderately large deflections of plates under uniformly distributed transverse load, the load parameter increases with the maximum deflection-to-thickness ratio. It is also noted from the post-buckling analysis results that the buckling load parameter increases with the deflection for the specified elastic foundation parameters. Nonlinear vibration analysis of initially stressed plates has been carried out considering the applied compressive load lower than the critical load of the plate and the applied uniformly distributed transverse load lower than 100th of load for the specified maximum amplitude. Tables 5 and 6 give the analysis results. It is noted that the frequency ratio (ω/ω_L) decreases with increase in the applied compressive load, whereas it increases with amplitude.

It should be noted that the solution of the problem is obtained by considering a single-mode transverse deflection function, which satisfies exactly the geometric boundary conditions. Utilizing this deflection function, the compatibility equation (12) is solved for the Airy stress function (φ), which exactly satisfies Eqs. (1) and (2). Using these functions in Eqs. (6)–(8) one can find many

Table 4
Nonlinear analysis results for a two-layered glass-epoxy cross-ply square plates on elastic foundations

Case	ς_s	Simply supported			Clamped		
		P/P_c Eq. (32)	Q_{NL}/Q_{LF} Eq. (33)	ω/ω_L Eq. (42)	P/P_c Eq. (32)	Q_{NL}/Q_{LF} Eq. (33)	ω/ω_L Eq. (42)
1	0.5	1.1318	0.5660	1.0481	1.0763	0.5382	1.0282
	1.0	1.5272	1.5272	1.1793	1.3054	1.3054	1.1078
	1.5	2.1863	3.2794	1.3683	1.6871	2.5306	1.2280
	2.0	3.1089	6.2179	1.5940	2.2215	4.4430	1.3776
2	0.5	1.1114	0.5556	1.0408	1.0729	0.5363	1.0269
	1.0	1.4456	1.4456	1.1536	1.2915	1.2915	1.1032
	1.5	2.0025	3.0038	1.3185	1.6558	2.4837	1.2186
	2.0	2.7823	5.5645	1.5181	2.1659	4.3317	1.3628
3	0.5	1.0441	0.5220	1.0164	1.0456	0.5227	1.0169
	1.0	1.1762	1.1762	1.0637	1.1822	1.1822	1.0658
	1.5	1.3965	2.0946	1.1378	1.4100	2.1151	1.1422
	2.0	1.7048	3.4095	1.2333	1.7290	3.4579	1.2404
4	0.5	1.1223	0.5612	1.0447	1.0759	0.5379	1.0280
	1.0	1.4891	1.4891	1.1674	1.3036	1.3036	1.1072
	1.5	2.1005	3.1508	1.3453	1.6830	2.5243	1.2268
	2.0	2.9565	5.9130	1.5591	2.2143	4.4281	1.3757
5	0.5	1.0301	0.5151	1.0112	1.0345	0.5173	1.0128
	1.0	1.1206	1.1206	1.0441	1.1381	1.1381	1.0503
	1.5	1.2712	1.9069	1.0964	1.3107	1.9660	1.1096
	2.0	1.4822	2.9644	1.1652	1.5523	3.1046	1.1871

Table 5
Frequency ratio (ω/ω_L) for the specified maximum amplitude-to-thickness ratio (ζ_+) of initially stressed two-layered glass-epoxy cross-ply square plates on elastic foundations without transverse distributed load ($Q_{NL}/Q_{LF} = 0$)

ζ_+	Compressive load ratio (P/P_c)				
	0.00	0.25	0.50	0.75	1.00
(a) *Simply supported square plates*					
Case 1: $K = K_1 = G = 0$					
0.5	1.0481	0.9211	0.7734	0.5896	0.3076
1.0	1.1793	1.0676	0.9424	0.7970	0.6152
1.5	1.3683	1.2726	1.1686	1.0537	0.9228
2.0	1.5940	1.5121	1.4250	1.3317	1.2303
Case 2: $K = 50$, $K_1 = 0$, $G = 0$					
0.5	1.0408	0.9128	0.7636	0.5768	0.2828
1.0	1.1536	1.0392	0.9103	0.7590	0.5655
1.5	1.3185	1.2191	1.1103	0.9889	0.8483
2.0	1.5181	1.4320	1.3400	1.2405	1.3110
Case 3: $K = 50$, $K_1 = 0$, $G = 25$					
0.5	1.0164	0.8849	0.7300	0.5319	0.1778
1.0	1.0637	0.9388	0.7944	0.6166	0.3556
1.5	1.1378	1.0217	0.8903	0.7352	0.5334
2.0	1.2333	1.1267	1.0086	0.8738	0.7113
Case 4: $K = 50$, $K_1 = 25$, $G = 0$					
0.5	1.0447	0.9172	0.7688	0.5837	0.2963
1.0	1.1674	1.0544	0.9275	0.7795	0.5925
1.5	1.3453	1.2479	1.1418	1.0240	0.8888
2.0	1.5591	1.4753	1.3860	1.2900	1.1851
Case 5: $K = 50$, $K_1 = 25$, $G = 50$					
0.5	1.0112	0.8790	0.7229	0.5220	0.1471
1.0	1.0441	0.9165	0.7680	0.5826	0.2942
1.5	1.0964	0.9755	0.8372	0.6705	0.4412
2.0	1.1652	1.0520	0.9248	0.7763	0.5883
(b) *Clamped square plates*					
Case 1: $K = K_1 = G = 0$					
0.5	1.0282	0.8984	0.7463	0.5539	0.2341
1.0	1.1078	0.9883	0.8521	0.6888	0.4682
1.5	1.2280	1.1210	1.0022	0.8664	0.7023
2.0	1.3776	1.2826	1.1795	1.0656	0.9364
Case 2: $K = 50$, $K_1 = 0$, $G = 0$					
0.5	1.0269	0.8969	0.7446	0.5516	0.2287
1.0	1.1032	0.9831	0.8461	0.6814	0.4574
1.5	1.2186	1.1107	0.9908	0.8533	0.6861
2.0	1.3628	1.2667	1.1623	1.0467	0.9148
Case 3: $K = 50$, $K_1 = 0$, $G = 25$					
0.5	1.0169	0.8855	0.7308	0.5329	0.1808
1.0	1.0658	0.9412	0.7972	0.6202	0.3617

Table 5 (*continued*)

ζ_+	Compressive load ratio (P/P_c)				
	0.00	0.25	0.50	0.75	1.00
1.5	1.1422	1.0265	0.8959	0.7419	0.5425
2.0	1.2404	1.1345	1.0172	0.8837	0.7233
Case 4: $K = 50,\ K_1 = 25,\ G = 0$					
0.5	1.0280	0.8982	0.7461	0.5536	0.2334
1.0	1.1072	0.9877	0.8513	0.6878	0.4668
1.5	1.2268	1.1196	1.0007	0.8647	0.7002
2.0	1.3757	1.2805	1.1773	1.0632	0.9336
Case 5: $K = 50,\ K_1 = 25,\ G = 50$					
0.5	1.0128	0.8808	0.7251	0.5252	0.1574
1.0	1.0503	0.9236	0.7764	0.5935	0.3148
1.5	1.1096	0.9903	0.8543	0.6916	0.4722
2.0	1.1871	1.0761	0.9520	0.8083	0.6296

Table 6
Frequency ratio (ω/ω_L) for the specified maximum amplitude-to-thickness ratio (ζ_+) of initially stressed ($P/P_c = 0.5$) two layered glass-epoxy cross-ply clamped square plates on elastic foundations with uniformly distributed transverse load

ζ_+	Simply supported			Clamped		
	ζ_-	$\frac{Q_{NL}}{Q_{LF}}$	$\frac{\omega}{\omega_L}$	ζ_-	$\frac{Q_{NL}}{Q_{LF}}$	$\frac{\omega}{\omega_L}$
Case 1: $K = K_1 = G = 0$						
0.5	−0.4820	0.0057	0.7712	−0.4810	0.0054	0.7450
1.0	−0.9700	0.0153	0.9364	−0.9670	0.0131	0.8479
1.5	−1.4600	0.0328	1.1591	−1.4570	0.0253	0.9952
2.0	−1.9510	0.0622	1.4123	−1.9470	0.0444	1.1698
Case 2: $K = 50,\ K_1 = 0,\ G = 0$						
0.5	−0.4820	0.0056	0.7617	−0.4810	0.0054	0.7433
1.0	−0.9690	0.0145	0.9049	−0.9670	0.0129	0.8420
1.5	−1.4590	0.0300	1.1017	−1.4560	0.0248	0.9839
2.0	−1.9500	0.0556	1.3283	−1.9470	0.0433	1.1529
Case 3: $K = 50,\ K_1 = 0,\ G = 25$						
0.5	−0.4810	0.0052	0.7292	−0.4810	0.0052	0.7300
1.0	−0.9650	0.0118	0.7916	−0.9650	0.0118	0.7943
1.5	−1.4530	0.0209	0.8854	−1.4530	0.0212	0.8908
2.0	−1.9420	0.0341	1.0014	−1.9430	0.0346	1.0100
Case 4: $K = 50,\ K_1 = 25,\ G = 0$						
0.5	−0.4820	0.0056	0.7668	−0.4810	0.0054	0.7448
1.0	−0.9690	0.0149	0.9218	−0.9670	0.0130	0.8471

Table 6 (*continued*)

ζ_+	Simply supported			Clamped		
	ζ_-	$\frac{Q_{NL}}{Q_{LF}}$	$\frac{\omega}{\omega_L}$	ζ_-	$\frac{Q_{NL}}{Q_{LF}}$	$\frac{\omega}{\omega_L}$
1.5	−1.4600	0.0315	1.1327	−1.4570	0.0252	0.9937
2.0	−1.9510	0.0591	1.3738	−1.9470	0.0443	1.1677
Case 5: $K = 50$, $K_1 = 25$, $G = 50$						
0.5	−0.4810	0.0052	0.7223	−0.4810	0.0052	0.7245
1.0	−0.9640	0.0112	0.7660	−0.9640	0.0114	0.7741
1.5	−1.4500	0.0191	0.8334	−1.4510	0.0197	0.8501
2.0	−1.9390	0.0296	0.9192	−1.9400	0.0310	0.9459

terms for strains (ε_x, ε_y, ε_{xy}) and the moment resultants (M_x, M_y, M_{xy}). The assumed transverse deflection function (w) and the corresponding strains (ε_x, ε_y, ε_{xy}) exactly satisfy the compatibility equation (10). The modal equation (23) is derived using these results in the equilibrium equation and applying the modified Galerkin's method. The results obtained from the exact integration of the modal equation are found to be reasonably in good agreement with the other existing approximate multi-mode solutions.

5. Concluding remarks

Studies are made on the elastic behavior of laminated thin square plates with moderately large deflection, post-buckling and nonlinear vibration. The effects of foundation parameters and edge conditions are examined. In all the cases considered, the nonlinear frequency increases with the amplitude and hardening type of nonlinearity is noted. Regarding the analysis for moderately large deflections of laminated square plates under uniformly distributed transverse load, the load parameter increases with the deflection. It is also noted from the post-buckling analysis results that the buckling load parameter increases with the deflection for the specified elastic foundations. Though numerical results presented in this paper are for square plates, the formulation is general and applicable for generally laminated thin rectangular plates.

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Appendix A

The expressions for the functions φ_1 and φ_2 in Eq. (20) are given below:

A.1. Simply supported rectangular plates

$$\varphi_1(x,y) = \psi_1 \sin \alpha_m x \sin \beta_n y + \psi_2 \cos \alpha_m x \cos \beta_n y,$$

$$\varphi_2(x,y) = \psi_3 \cos 2\alpha_m x + \psi_4 \cos 2\beta_n y,$$

$$\psi_1 = (\psi_{11}\psi_{21} - \psi_{12}\psi_{22})/(\psi_{11}^2 - \psi_{12}^2), \quad \psi_2 = (\psi_{11}\psi_{22} - \psi_{12}\psi_{21})/(\psi_{11}^2 - \psi_{12}^2),$$

$$\psi_3 = \beta_n^2/(32A_{22}^*\alpha_m^2), \quad \psi_4 = \alpha_m^2/(32A_{11}^*\beta_n^2),$$

$$\psi_{11} = \alpha_m^4 A_{22}^* + \alpha_m^2\beta_n^2(2A_{12}^* + A_{66}^*) + \beta_n^4 A_{11}^*, \quad \psi_{12} = 2\alpha_m^3\beta_n A_{26}^* + 2\alpha_m\beta_n^3 A_{16}^*,$$

$$\psi_{21} = \alpha_m^4 B_{21}^* + \alpha_m^2\beta_n^2(B_{11}^* + B_{22}^* - 2B_{66}^*) + \beta_n^4 B_{12}^*, \quad \psi_{22} = \alpha_m^3\beta_n(B_{61}^* - 2B_{26}^*) + \alpha_m\beta_n^3(B_{62}^* - 2B_{16}^*).$$

A.2. Clamped rectangular plates

$$\varphi_1(x,y) = \psi_1 \cos 2\alpha_m x + \psi_2 \cos 2\beta_n y + \psi_3 \cos 2\alpha_m x \cos 2\beta_n y + \psi_4 \sin 2\alpha_m x \sin 2\beta_n y,$$

$$\begin{aligned}\varphi_2(x,y) = {} & \psi_5 \cos 2\alpha_m x + \psi_6 \cos 2\beta_n y + \psi_7 \cos 4\alpha_m x + \psi_8 \cos 4\beta_n y \\ & + \psi_9 \cos 2\alpha_m x \cos 2\beta_n y + \psi_{10} \sin 2\alpha_m x \sin 2\beta_n y + \psi_{11} \cos 2\alpha_m x \cos 4\beta_n y \\ & + \psi_{12} \sin 2\alpha_m x \sin 4\beta_n y + \psi_{13} \cos 4\alpha_m x \cos 2\beta_n y + \psi_{14} \sin 4\alpha_m x \sin 2\beta_n y,\end{aligned}$$

$$\psi_1 = -\frac{B_{21}^*}{4A_{22}^*}, \quad \psi_2 = -\frac{B_{12}^*}{4A_{11}^*},$$

$$\psi_3 = \frac{\psi_{31}\psi_{41} - \psi_{32}\psi_{42}}{4(\psi_{31}^2 - \psi_{32}^2)}, \quad \psi_4 = \frac{\psi_{31}\psi_{42} - \psi_{32}\psi_{41}}{4(\psi_{31}^2 - \psi_{32}^2)}, \quad \psi_5 = \frac{1}{32}\left(\frac{\beta_n^2}{A_{22}^*\alpha_m^2}\right),$$

$$\psi_6 = \frac{1}{32}\left(\frac{\alpha_m^2}{A_{11}^*\beta_n^2}\right), \quad \psi_7 = \frac{-1}{512}\left(\frac{\beta_n^2}{A_{22}^*\alpha_m^2}\right), \quad \psi_8 = \frac{-1}{512}\left(\frac{\alpha_m^2}{A_{11}^*\beta_n^2}\right),$$

$$\psi_9 = \frac{-1}{16}\left(\frac{\psi_{31}\alpha_m^2\beta_n^2}{\psi_{31}^2 - \psi_{32}^2}\right), \quad \psi_{10} = \frac{1}{16}\left(\frac{\psi_{32}\alpha_m^2\beta_n^2}{\psi_{31}^2 - \psi_{32}^2}\right), \quad \psi_{11} = \frac{1}{32}\left(\frac{\psi_{71}\alpha_m^2\beta_n^2}{\psi_{71}^2 - \psi_{72}^2}\right),$$

$$\psi_{12} = \frac{-1}{32}\left(\frac{\psi_{72}\alpha_m^2\beta_n^2}{\psi_{71}^2 - \psi_{72}^2}\right), \quad \psi_{13} = \frac{1}{32}\left(\frac{\psi_{81}\alpha_m^2\beta_n^2}{\psi_{81}^2 - \psi_{82}^2}\right), \quad \psi_{14} = \frac{-1}{32}\left(\frac{\psi_{82}\alpha_m^2\beta_n^2}{\psi_{81}^2 - \psi_{82}^2}\right),$$

$$\psi_{31} = \alpha_m^4 A_{22}^* + \alpha_m^2\beta_n^2(2A_{12}^* + A_{66}^*) + \beta_n^4 A_{11}^*, \quad \psi_{32} = 2\alpha_m^3\beta_n A_{26}^* + 2\alpha_m\beta_n^3 A_{16}^*,$$

$$\psi_{41} = \alpha_m^4 B_{21}^* + \alpha_m^2\beta_n^2(B_{11}^* + B_{22}^* - 2B_{66}^*) + \beta_n^4\beta_{12}^*, \quad \psi_{42} = \alpha_m^3\beta_n(B_{61}^* - 2B_{26}^*) + \alpha_m\beta_n^3(B_{62}^* - 2B_{16}^*),$$

$$\psi_{71} = \alpha_m^4 A_{22}^* + 4\alpha_m^2\beta_n^2(2A_{12}^* + A_{66}^*) + 16\beta_n^4 A_{11}^*, \quad \psi_{72} = 4\alpha_m^3\beta_n A_{26}^* + 16\alpha_m\beta_n^3,$$

$$\psi_{81} = 16\alpha_m^4 A_{22}^* + 4\alpha_m^2\beta_n^2(2A_{12}^* + A_{66}^*) + \beta_n^4 A_{11}^*, \quad \psi_{82} = 16\alpha_m^3\beta_n A_{26}^* + 4\alpha_m\beta_n^3 A_{16}^*.$$

Appendix B

The constants α_p, β, γ, and δ in the equation of motion (24) are given below:

B.1. Simply supported rectangular plates

$$\alpha = \psi_0 + \psi_{01} + k + g(\alpha_m^2 + \beta_n^2), \quad \alpha_p = \frac{P_x\alpha_m^2}{b} + \frac{P_y\beta_m^2}{a}.$$

$$\text{For } m \text{ and } n \text{ are odd}: \quad \beta = -\left(\frac{8\alpha_m\beta_n}{3ab}\right)\left(4\psi_1 + \frac{B_{21}^*}{A_{22}^*} + \frac{B_{12}^*}{A_{11}^*}\right).$$

$$\text{For } m \text{ is even and } n \text{ is odd}: \quad \beta = \left(-\frac{2\alpha_m\beta_n}{ab}\right)\left(\frac{-B_{21}^*}{A_{22}^*} + \frac{\alpha_m^2 B_{11}^*}{3\beta_n^2 A_{11}^*}\right).$$

$$\text{For } m \text{ is odd and } n \text{ is even}: \quad \beta = \left(-\frac{2\alpha_m\beta_n}{ab}\right)\left(\frac{-B_{12}^*}{A_{11}^*} + \frac{\beta_n^2 B_{22}^*}{3\alpha_m^2 A_{22}^*}\right).$$

$$\text{For } m \text{ and } n \text{ are even}: \quad \beta = 0.$$

$$\delta = \frac{16}{mn\pi^2}\sin^2\frac{m\pi}{2}\sin^2\frac{n\pi}{2}, \quad \gamma = \frac{1}{16}\left(\frac{\alpha_m^4}{A_{11}^*} + \frac{\beta_n^4}{A_{22}^*}\right) + \frac{9}{16}k_1,$$

$$\psi_0 = \alpha_m^4 D_{11}^* + 2\alpha_m^2\beta_n^2(D_{12}^* + 2D_{66}^*) + \beta_n^4 D_{22}^*, \quad \psi_{01} = \frac{\psi_{11}(\psi_{21}^2 + \psi_{22}^2) - 2\psi_{12}\psi_{21}\psi_{22}}{\psi_{11}^2 - \psi_{12}^2}.$$

B.2. Clamped rectangular plates

$$\alpha = \frac{16}{9}\left[\psi_0 + \psi_{01} + \frac{2B_{21}^{*2}\alpha_m^4}{A_{22}^*} + \frac{2B_{12}^{*2}\beta_n^4}{A_{11}^*}\right] + k + \tfrac{4}{3}g(\alpha_m^3 + \beta_n^2),$$

$$\alpha_p = \frac{3}{4}\left(\frac{P_x\alpha_m^2}{b} + \frac{P_y\beta_m^2}{a}\right), \quad \beta = \frac{-4}{3}(\alpha_m^2\beta_n^2)\left(\psi_{02} + \frac{B_{21}^*}{A_{22}^*} + \frac{B_{12}^*}{A_{11}^*}\right),$$

$$\gamma = \frac{17}{144}\left[\frac{\alpha_m^4}{A_{11}^*} + \frac{\beta_n^4}{A_{22}^*}\right] + \frac{2}{9}\left(\frac{\alpha_m^4\beta_n^4\psi_{02}}{\psi_{02}^2 - \psi_{03}^2}\right) + \frac{1}{18}\left(\frac{\alpha_m^4\beta_n^4\psi_{71}}{\psi_{71}^2 - \psi_{72}^2}\right) + \frac{1}{18}\left(\frac{\alpha_m^4\beta_n^4\psi_{81}}{\psi_{81}^2 - \psi_{82}^2}\right) + \frac{1225}{2304}k_1,$$

$$\delta = \frac{16}{9}, \quad \psi_0 = 3\alpha_m^4 D_{11}^* + 2\alpha_m^2\beta_n^2(D_{12}^* + 2D_{66}^*) + 3\beta_n^4 D_{22}^*,$$

$$\psi_{01} = \frac{\psi_{31}(\psi_{41}^2 + \psi_{42}^2) - 2\psi_{32}\psi_{41}\psi_{42}}{\psi_{31}^2 - \psi_{32}^2}, \quad \psi_{02} = \frac{\psi_{31}\psi_{41} - \psi_{32}\psi_{42}}{\psi_{31}^2 - \psi_{32}^2}.$$

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